Formal Concept Analysis and Fuzzy Logic

Radim BELOHLAVEK

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Palacky University, Olomouc
Czech Republic

Outline

aims

- overview of recent work on concept lattices, attribute implications, and related structures and topics in fuzzy setting
- attention paid to one particular approach, but several other approaches are overviewed
- present future directions

content

- PART I: Introduction to fuzzy logic
- PART II: Concept lattices and related structures in fuzzy setting
- PART III: Attribute implications in fuzzy setting

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PART I

An Introduction to Fuzzy Logic

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- motivation, brief historical overview
- structures of truth degrees
- basics of fuzzy sets
- fuzzy logic as logic

Basic idea of fuzzy logic

natural language expression like "normal blood pressure"
how to represent its meaning?
extensionally (common way in knowledge representation):
collection of values of "normal blood pressure"

"normal blood pressure"
as (ordinary) set

vs.

"normal blood pressure"
as **fuzzy set**

– technically: **replace $\{0,1\}$ by $[0,1]$**,
i.e. degrees of truth instead of false and true
– **fuzzy approach** (graded approach) $\Rightarrow$ "paradigm shift"
– proposed by Lotfi A. Zadeh (1965)

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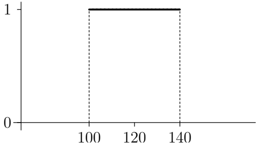

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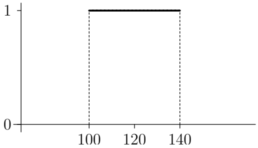

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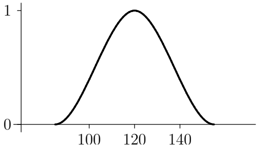

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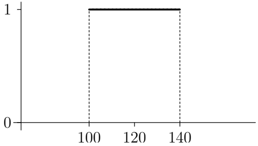

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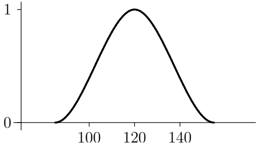

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Graded approach to modeling

- empirical relationships, properties, and groupings are vague
- examples: x loves y, x is similar to y, x is satisfied, ...
- modeling vague relationships, groupings, etc. using ordinary relations, sets, etc., is not appropriate
- vague relationships cannot be avoided (they are being used in everyday communication)
- lack of mathematical theory for dealing with vague relationships recognized by several scientists long time ago (before Zadeh)
  - Bertrand Russell: Vagueness. Austral. J. Philos. **1**(1923), 84–92.
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Black (Vagueness: an exercise in logical analysis, 1937):

"It is a paradox that the most developed and useful theories are expressed in terms of objects never encountered in experience. While the mathematician constructs a theory in terms of "perfect" objects, the experimental scientist observes of which the properties demanded by theory are only approximately true. As Duhem remarks, mathematical deduction is not useful to the physicist if interpreted rigorously. It is necessary to know that its validity is unaltered when the premise and conclusion are only "approximately true". But the indeterminacy thus introduced, it is necessary to add in criticism, will invalidate the deduction unless the permissible limits of variation are specified. **To do so, however, replaces the original mathematical deduction by a more complicated mathematical theory whose exact nature is in any case unknown.**"

Zadeh (Fuzzy sets, 1965):

"More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. ... Clearly, the "class of all real numbers which are much greater than 1", or "the class of beautiful women" ..., or "the class of tall men", **do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.**"

Historical development of fuzzy logic and fuzzy sets

- Black (1937): "consistency profiles"
- Zadeh (1965): fuzzy set
  - fuzzy set in universe X: mapping $A : X \to [0,1]$
  - $A(x) \in [0,1]$ ... degree to which x belongs to A
  - calculus of fuzzy sets
- other important contributions in 1960s and 1970s (R. Bellman, J. A. Goguen, ...)
- E. H. Mamdani, S. Assilian: An experiment in linguistic synthesis with a fuzzy logic controller. Int. J. Man-Machine Studies 7(1975), 1–13.
- fuzzy controllers = commercially most successful applications of FL
  - fuzzy washing machines, fuzzy cameras, car transmission systems, ABS systems, metro in Japan cities, ..., unmanned helicopter
  - boom in Japan (consumer electronics) 1990s, according to MITI: 1991 world computer market (SW, HW, services) = \$200bn; 1991 market with "fuzzy products" = \$2bn;

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  - structures of truth degrees, fuzzy relations, foundational aspects, etc.
  - fuzzy control, approximate reasoning, information retrieval, decision making, pattern recognition, etc.
- fuzzy logic as logic?: not many contributions prior to 1990s
- late 1990s and 2000s: formal systems of fuzzy logic (plus applications to various domains, theoretical and application-oriented)

further issues

- contributions to both theory and applications of fuzzy logic often *ad hoc* (idea of FL appealing but ...)
- confusion with probability: fuzziness (vagueness) $\neq$ randomness
- conferences: IFSA, FUZZ-IEEE, NAFIPS, ...
- journals: Fuzzy Sets and Systems, IEEE Trans. Fuzzy Systems, Int. J. Approx. Reasoning, Soft Computing, IJUFKBS ...

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Recent statistics

- patents issued (fuzzy-logic-related)
  - Japan: 4,801
  - US: around 1,700
- journals ("fuzzy" in title)
  - 12
- publications ("fuzzy" in title, INSPEC database)

1970–1979	521
1980-1989	2,163
1990-1999	20,210
2000-present	48,627
total	71,521

- citations to papers by Zadeh
  - 28,122 (Web of Science)
  - 85,500 (Google Scholar)

"Technical aspects" of fuzzy logic

basic principle: allow propositions to have **intermediate truth degrees** instead of just 0 (false) and 1 (true), e.g.

$$||\text{John is tall}|| = 0.9$$
$$||x \text{ belongs to } A|| = 0.7$$

as in classical case, fuzzy logic is **truth functional**, i.e.
truth degrees of complex propositions are computed from tr. deg. of constituent propositions:

$$||\ \varphi \text{ AND } \psi\ || = ||\ \varphi\ || \otimes ||\ \psi\ ||$$

where $\otimes$ is a (truth function of) conjunction connective, i.e.
$\otimes : [0,1] \times [0,1] \to [0,1]$
this brings us to

- what sets of truth degrees shall we use?
- what connectives shall we use?

i.e. what **structures of truth degrees** for fuzzy logic?

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Structures of truth degrees

classical logic: two-element Boolean algebra, given by

- set $\{0, 1\}$ of truth degrees,
- (truth functions of) logical connectives (conjunction, implication, . . .)

fuzzy logic: several possibilities proposed

- (partially ordered) set L of truth degrees, e.g. $L = [0, 1]$, $L = \{0, 0.1, 0.2, \ldots, 1\}$, non-linearly ordered L, . . .
- (truth functions of) logical connectives (conj. $\otimes$, impl. $\rightarrow$, . . .)

reasonable approach: instead of working with a fixed set of truth degrees and fixed logical connectives

– postulate properties of sets of truth degrees and logical connectives

– any structure of truth degrees satisfying the properties is OK

– leave the choice of particular set and connectives to further circumstances (e.g., complexity considerations)

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Structures of truth degrees: residuated lattices

Complete residuated lattice – basic structure of truth degrees

$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, where

- $\langle L, \wedge, \vee, 0, 1 \rangle$ ... complete lattice,
- $\langle L, \otimes, 1 \rangle$ ... commutative monoid,
- $\langle \otimes, \rightarrow \rangle$ ... **adjoint** pair (i.e., $a \otimes b \leq c$ iff $a \leq b \rightarrow c$).

Remarks

- introduced by Dilworth and Ward (Trans. AMS, 1939)
- proposed as structure of truth degrees by Goguen (Synthese, 1969)
- $a \in L$ ... truth degrees
- infs and sups needed for evaluation of quantifiers
- commutativity of $\otimes$: we want $||\varphi \otimes \psi|| = ||\psi \otimes \varphi||$
- adjointness results from considerations about *modus ponens* (Goguen)

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Examples residuated lattices

Structures on $[0,1]$

$\mathbf{L} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ given by left-continuous (continuous) $\otimes$.

- **Łukasiewicz**: $a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$.
- **Gödel (minimum)**: $a \otimes b = \min(a,b)$, $a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$
- **Goguen (product)**: $a \otimes b = a \cdot b$, $a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$

Finite structures of truth degrees

$L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ... finite subset of $[0,1]$

finite Łukasiewicz chain / Gödel chain ... restrictions of $\otimes, \rightarrow$ on finite L

Important classes of residuated lattices

BL-algebras, Heyting algebras, MV-algebras, Boolean algebras

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Hedges: examples of additional connectives

studied by Takeuti+Titani, Baaz, Hájek, ...

(idempotent) truth-stressing **hedge** ... mapping $^*\colon L \to L$ satisfying

$$1^* = 1, \qquad a^* \leq a, \qquad (a \to b)^* \leq a^* \to b^*, \qquad a^{**} = a^*,$$

meaning of *: truth function of logical connective "very true"

Two boundary hedges

- **identity**, i.e. $a^* = a$ $(a \in L)$;
- **globalization**: $a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$ ... truth function of "fully true"

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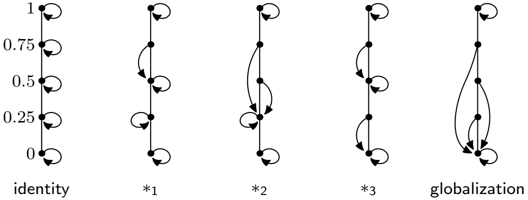

Fuzzy sets and fuzzy relations

$\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, \otimes, ^*, 0, 1 \rangle$ ... residuated structure of truth degrees

Fuzzy sets (L-sets)

L-set A in universe U ... mapping $A\colon U \rightarrow L$,
interpretation of $A(u)$: "degree to which u belongs to A",
for U finite: $A = \{\ldots, {}^{A(u)}/u, \ldots\}$

Fuzzy relations (L-relations)

binary L-relation R between U and V ... mapping $R\colon U \times V \rightarrow L$,
$R(u, v)$: "degree to which $u \in U$ and $v \in V$ are R-related"

Operations with L-sets

$\mathbf{L}^U$ ... collection of all **L**-sets in universe U,
operations are defined componentwise, e.g. $(A \cup B)(u) = A(u) \vee B(u)$, ...

Inclusion

$A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$ ... as ordinary relation
$S(A, B) = \bigwedge_{u \in U}(A(u) \rightarrow B(u))$ ... as fuzzy relation

$$A \subseteq B \text{ iff } S(A, B) = 1$$

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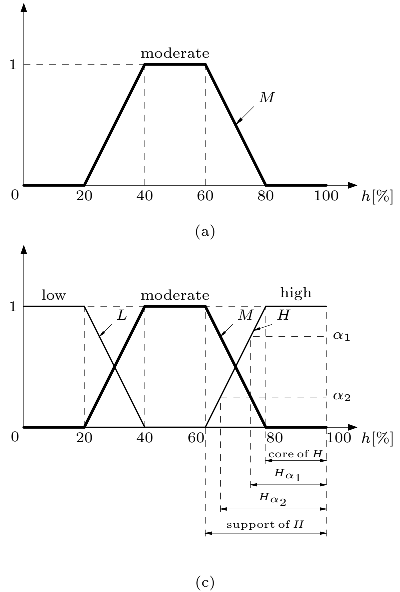
Fuzzy sets representing humidity concepts



Structure L of truth degrees as parameter

when approaching something (sets, equivalence relations, attribute implications) from FL point of view, work with **L** as with a parameter

example: equivalence relations

- a single theory of **L**-equivalence relations (uses only general properties of residuated lattices **L**); **L** is a parameter;
- particular choices of **L** lead to particular theories of fuzzy equivalence relations (covers both finite and infinite sets of truth degrees, etc.)
- ordinary equivalence relations = fuzzy equivalence relations when $\mathbf{L} = \mathbf{2}$ (two-element Bool. algebra)

in particular: **ordinary sets = fuzzy sets for $\mathbf{L} = \mathbf{2}$**
(convention: we identify sets with their characteristic functions)

Useful general view.
As opposed to the view when fuzzy set means $L = [0, 1]$.

Structure L of truth degrees as parameter

when approaching something (sets, equivalence relations, attribute implications) from FL point of view, work with **L** as with a parameter

example: equivalence relations

- a single theory of **L**-equivalence relations (uses only general properties of residuated lattices **L**); **L** is a parameter;
- particular choices of **L** lead to particular theories of fuzzy equivalence relations (covers both finite and infinite sets of truth degrees, etc.)
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Symbolic and numerical level of FL

- **symbolic level**: analogous to classical logic
  (symbolic formulas, can be read in natural language, clear meaning)
- **numerical level**: missing (trivial) in classical logic
  (formulas have numerical truth values, these are manipulated using fuzzy logic connnectives, quantitative information)

Example (transitivity of relation R):

ordinary setting:
"if $\langle x, y\rangle$ is in R and $\langle y, z\rangle$ is in R then $\langle x, z\rangle$ is in R", i.e.
$||r_R(x, y) \;\&\; r_R(y, z) \Rightarrow r_R(x, z)||=1$ (formula is true when $r_R \mapsto R$)

fuzzy setting:
symbolic level: same formula as in ordinary setting, i.e. same meaning
numerical level: the formula for transitivity is true iff

$$R(x, y) \otimes R(y, z) \leq R(x, z), \qquad \text{e.g. } 0.7 \otimes 0.8 \leq R(x, z).$$

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Fuzzy logic as logic

Is there any logic (study of consequence, provability, etc.) in fuzzy logic?

terminological distinction (Zadeh)

- fuzzy logic in narrow sense: mathematical logic with degrees of truth
- fuzzy logic in broad sense: any application of fuzzy approach

fuzzy logic substantially developed by Hájek in late 1990s

many other prominent logicians since then (Parikh, Mundici, Gabbay, ..., Baaz, Ono, ...)

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Ordinary style fuzzy logic

- agenda analogous to agenda of classical logic (propositional, predicate, etc.)
- difference: intermediate truth degrees in addition to 0 and 1

basics:

- formulas φ, e.g. $(\forall x, y)(r(x, y) \Rightarrow r(y, x))$, $A \Rightarrow B$, etc.
- semantic structures $\mathbf{M}$ (first-order structures, fuzzy contexts, etc.)
- assignment of truth degrees: $||\varphi||_{\mathbf{M}} \in L$; L is support of $\mathbf{L}$
- truth functionality: $||\varphi \Rightarrow \phi||_{\mathbf{M}} = ||\varphi||_{\mathbf{M}} \rightarrow ||\phi||_{\mathbf{M}}$
- theory T: a set of formulas
- models of theories: $\mathrm{Mod}(T) = \{\mathbf{M} \mid \text{for each } \varphi \in T : ||\varphi||_{\mathbf{M}} = 1\}$
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- approach due to Goguen (1960s), Pavelka (1970s), further developed by Novak, Hajek, Gerla, . . . , RB, Vychodil
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- difference: **interest in degrees**
  - to which formula is present in theory,
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Ordinary and Pavelka style fuzzy logic

- reasonable way to approach reasoning under uncertainty
- Pavelka style: general framework, including non-truth-functional semantics (reasoning about probability, Gerla)
- provides methodology for formal treatment of reasoning in particular domains (e.g., attribute implications)

Textbooks and monographs

Textbooks

- Dubois, Prade: Fuzzy Sets and Systems: Theory and Applications. Academic Press, 1982.
- Klir, Yuan: Fuzzy Sets and Fuzzy Logic. Theory and Applications. Prentice Hall, 1995.
- Nguyen, Walker: A First Course in Fuzzy Logic. Chapman & Hall/CRC Press, 2000.

Monographs

- Hájek: Metamathematics of Fuzzy Logic. Kluwer, 1998.
- Gottwald: A Treatise on Many-Val. Logics. Res. Stud. Press, 2001.
- Gerla: Fuzzy Logic. Math. Tools for Appr. Reasoning. Kluwer, 2001.
- Novák, Perfilieva, Močkoř: Mathematical Principles of Fuzzy Logic. Kluwer, 1999.
- Belohlavek: Fuzzy Relational Systems: Foundations and Principles. Kluwer, 2002.

Four (personal) remarks on fuzzy logic

1. FL appealing, interesting, useful, non-trivial.

2. Caution: huge number of very-low quality papers.

3. Criticisms of FL (many of them wrong, based on technical or conceptual flaws; some of them very good and useful).

4. Technically highly developed in some respects (algebraic and logical aspects) but severely underdeveloped in other respects (e.g. connections to mathematical psychology).