Relations between Proto-fuzzy Concepts, Crisply Generated Fuzzy Concepts, and Interval Pattern Structures

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Recently several models for the analysis of fuzzy and numerical data within FCA were proposed:

- Fuzzy formal concepts [R. Bělohlávek, 1999]
- Protofuzzy formal concepts [O. Krídlo, S. Krajčí, 2005]
- Pattern structures [B. Ganter and S.O. Kuznetsov, 2000]
Main research goal

Relationships between protofuzzy concepts, fuzzy concepts, pattern structures, both from theoretical and from experimental viewpoints.
Outline

Relationships between

- Protofuzzy [O. Krídlo, S. Krajči] and fuzzy formal concepts [R. Bělohlávek]
- Fuzzy formal concepts [R. Bělohlávek] and pattern structures [B. Ganter and S.O. Kuznetsov]
- Fuzzy formal concepts [R. Bělohlávek] and two-way pattern structures
Main definitions of FCA

$G$ is a set, called the set of objects, $M$ is a set, called the set of attributes, $I \subseteq G \times M$ is a binary relation.

The triple $K := (G, M, I)$ is called a (formal) context.

**Galois connection** between $(2^G, \subseteq)$ and $(2^M, \subseteq)$ is a pair of maps

\[
(\cdot)' : 2^G \rightarrow 2^M \quad A' \overset{\text{def}}{=} \{ m \in M \mid gIm \text{ for all } g \in A \},
\]

\[
(\cdot)' : 2^M \rightarrow 2^G \quad B' \overset{\text{def}}{=} \{ g \in G \mid glm \text{ for all } m \in B \},
\]

with the following properties:

$\forall A_1, A_2 \subseteq G, B_1, B_2 \subseteq M$

1. $A_1 \subseteq A_2 \Rightarrow A'_2 \subseteq A'_1$; \hspace{1cm} $B_1 \subseteq B_2 \Rightarrow B'_2 \subseteq B'_1$;

2. $A_1 \subseteq A''_1$, \hspace{1cm} $B_1 \subseteq B''_1$. 
Main definitions of FCA

Formal concepts are partially ordered by the relation
\[ \langle A_1, B_1 \rangle \geq \langle A_2, B_2 \rangle \iff A_1 \supseteq A_2 \quad B_2 \supseteq B_1. \]

**Implication** $A \rightarrow B$, where $A, B \subseteq M$, holds in the context if $A' \subseteq B'$. 

The map $(\cdot)''$ is a closure operator, since it is idempotent, extensive, and monotone.

**A formal concept** is a pair $\langle A, B \rangle$, $A \subseteq G$, $B \subseteq M$, such that $A' = B$, $B' = A$. 
A fuzzy formal context is a triple $\langle X, Y, I \rangle$, where $I$ is a fuzzy relation between $X$ and $Y$, taking a pair $(x, y)$ to the truth degree $I(x, y) \in L$, with which object $x \in X$ has attribute $y \in Y$.

A fuzzy formal concept is a pair $\langle A, B \rangle$, $A \subseteq X$, $B \subseteq Y$, such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$, where

$$A^{\uparrow}(y) = \bigwedge_{x \in X} (A(x) \rightarrow I(x, y))$$

and

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} (B(y) \rightarrow I(x, y)).$$

Example

$I = \begin{pmatrix} 0.1 & 0.6 & 0.5 \\ 1 & 0.2 & 0.4 \\ 0.8 & 0.3 & 0.0 \end{pmatrix}$

$A = (0.3, 0.9, 1)$ is the fuzzy tuple of extent

$B = (0.8, 0.3, 0)$ is the fuzzy tuple of intent
A pair \( \langle A, B \rangle \) such that \( A = A_c^{\uparrow \downarrow} = B^\downarrow \) and \( B = A^\uparrow = A_c^{\uparrow \downarrow \uparrow} \) is called a **crisply generated** formal concept (\( A_c \) is a crisp tuple).

\[
I = \begin{pmatrix}
0.1 & 0.6 & 0.5 \\
1 & 0.2 & 0.4 \\
0.8 & 0.3 & 0
\end{pmatrix}
\]

\( A_c = (0, 1, 0) \) is the crisp tuple of extent;

\[
A = (0.1, 1, 0.6) = B^\downarrow = A_c^{\uparrow \downarrow}; \quad B = (1, 0.2, 0.4) = A^\uparrow;
\]

\( \langle A, B \rangle \) is a crisply generated formal concept.

\( A_c = 1 A_c^{\uparrow \downarrow} \) is a tuple closed by crisp components (crisply closed tuple).
Protofuzzy formal concepts [O. Krídlo, S. Krajči]

Consider mappings \( \uparrow^l : 2^X \rightarrow 2^Y \) and \( \downarrow^l : 2^Y \rightarrow 2^X \) defined for a fuzzy context \( \langle X, Y, I \rangle \) over lattice \( L \), and \( l \in L \).

\[ \forall A \subseteq X \text{ let } \uparrow_l (A) = \{ y \in Y : (\forall x \in A) I(x, y) \geq l \} ; \]
\[ \forall B \subseteq Y \text{ let } \downarrow_l (B) = \{ x \in X : (\forall y \in B) I(x, y) \geq l \} . \]

**Definition**

An \( l \)-concept is a pair \( \langle A, B \rangle \) such that \( \uparrow_l (A) = B \) and \( \downarrow_l (B) = A \), i.e., a concept in the \( l \)-cut, binary context \( \langle X, Y, I_l \rangle \), where

\[ I_l = \{ (x, y) \in X \times Y : I(x, y) \geq l \} . \]

By \( \mathcal{K}_l \) we denote the set of all concepts in the \( l \)-cut.

**Definition**

Triples \( \langle A, B, l \rangle \in 2^X \times 2^Y \times L \) such that \( \langle A, B \rangle \in \bigcup_{k \in L} \mathcal{K}_k \) and \( l = \sup \{ k \in L : \langle A, B \rangle \in \mathcal{K}_k \} \) are called protofuzzy concepts.

The set of all protofuzzy concepts is denoted by \( \mathcal{K}^P \).
A protofuzzy formal concept is a formal concept that occurs for the “first time” in a $\gamma$-cut of a fuzzy context.

**Example**

$$ l = \begin{array}{ccc}
0.1 & 0.6 & 0.5 \\
1 & 0.2 & 0.4 \\
0.8 & 0.3 & 0.0 \\
\end{array} $$
A protofuzzy formal concept is a formal concept that occurs for the “first time” in a $\gamma$-cut of a fuzzy context.

<table>
<thead>
<tr>
<th></th>
<th>0.1</th>
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<td>0.8</td>
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</table>

0.5-cut
A protofuzzy formal concept is a formal concept that occurs for the “first time” in a $\gamma$-cut of a fuzzy context.

**Example**

\[
I = \begin{array}{ccc}
0.1 & 0.6 & 0.5 \\
1 & 0.2 & 0.4 \\
0.8 & 0.3 & 0
\end{array}
\]

\[
\begin{array}{cc}
X & X \\
0.5\text{-cut}
\end{array}
\]
Let $\mathcal{K}^P$ denote the set of all proto-fuzzy concepts.

**Definition**

Let $B \subseteq Y$ be an arbitrary set of attributes. The **contraction** of the set of proto-fuzzy concepts subsistent to the set $B$ is

$$\mathcal{K}_B^P = \{ \langle A, l \rangle \in 2^X \times L : (\exists B' \supseteq B)\langle A, B', l \rangle \in \mathcal{K}^P \}.$$
Relationship between protofuzzy formal concepts and fuzzy formal concepts

Theorem

Let there be a protofuzzy formal concept in the $l$-cut. Then there is a tuple $B_c$ that satisfies the following conditions:

1. $B_c = (a_1, \ldots, a_k)$, where $a_i \geq l$ if the $i$th object is contained in the given protofuzzy concept, and $a_i < l$ if it is not;

2. $B_c$ is a crisply closed tuple, i.e., $1_{B_c}^{\downarrow\uparrow} = B_c$.

Theorem

Let $A = (a_1, \ldots, a_k) = B_c^{\downarrow}$, $B = B_c^{\downarrow\uparrow}$. Let us define a tuple $Z$ by the formula

$$z_j = \begin{cases} 1, & a_j \geq a_i, \\ 0, & a_j < a_i. \end{cases}$$

Then the context $Z \times 1^a_Y$ corresponds to a protofuzzy concept in the $a_i$-cut of the initial context, which can be contracted to $1^a_Y$. 
Relationship between protofuzzy formal concepts and fuzzy concepts

Results:

- It is shown that for every protofuzzy concept one can construct a crisply closed tuple;
- It is shown that for every crisply-closed tuple one can construct a contraction of a protofuzzy concept on the crisp subset of attributes of its intent (this contraction may coincide with the intent itself);
- There is a method for deciding whether the contraction proposed is an intent of a protofuzzy concept.
Let $G$ be a set, called the set of objects, $(D, \sqcap)$ be a lower semilattice and $\delta : G \to D$ be a mapping.

**Definition**

The triple $(G, D, \delta)$, where $D = (D, \sqcap)$, is called a pattern structure if the set $\delta(G) := \{ \delta(g) \mid g \in G \}$ generates a complete subsemilattice $(D_\delta, \sqcap)$ of $(D, \sqcap)$.

For a pattern structure $(G, D, \delta)$ we define operations

$$A \Box := \bigsqcap_{g \in A} \delta(g)$$

for sets $A \subseteq G$ and

$$\text{d} \Box := \{ g \in G \mid \text{d} \sqsubseteq \delta(g) \}$$

for elements of semilattice $\text{d} \in D$.

A pattern concept is a pair $\langle A, \text{d} \rangle$ such that $A \Box = \text{d}$, $\text{d} \Box = A$. 
Interval pattern structures

<table>
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<tr>
<th></th>
<th>object 1</th>
<th>object 2</th>
<th>object 3</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>0.2</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.3</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

$\{1, 2\} = ([0.1, 1], [0.2, 0.6], [0.4, 0.5])$;

$([0.1, 1], [0.2, 0.6], [0.4, 0.5]) = \{1, 2\}$.

The pair $\{1, 2\} , ([0.1, 1], [0.2, 0.6], [0.4, 0.5])$ is an interval pattern concept.
Relationship between fuzzy formal concepts and pattern structures

Theorem

*If extents of interval pattern concepts coincide with crisply closed tuples, there are no nontrivial implications in the context.*

Theorem

*For any crisply-closed tuple the corresponding subset of objects is closed.*

Theorem

*If the set of crisply closed tuples coincides with the set of extents of interval pattern concepts, each interval pattern concept should satisfy the following condition: for no object from its extent the object intent majorizes the minimum of intervals.*
Two-way pattern structures

Let \((D_1, \sqcap_1, \sqcup_1)\) and \((D_2, \sqcap_2, \sqcup_2)\) be two arbitrary lattices with operations of infimum \(\sqcap_1, \sqcap_2\) and supremum \(\sqcup_1, \sqcup_2\), and a pair of mappings be given: \(\delta_1 : D_1 \rightarrow 2^{D_2}\) and \(\delta_2 : D_2 \rightarrow 2^{D_1}\).

We introduce operations \(\Box : D_1 \rightarrow D_2\) and \(\Box : D_2 \rightarrow D_1\) as follows:
- for \(a \in D_1\) we put \(a \Box = \sqcap_2\delta_1(a)\);
- for \(b \in D_2\) we put \(b \Box = \sqcap_1\delta_2(b)\).

The mapping \(\Box\Box\) is a closure operator (both over \(D_1\) and \(D_2\)).

Two-way pattern concepts are pairs \(\langle a, b \rangle\), \(a \in D_1, b \in D_2\), such that \(a \Box = b\) and \(b \Box = a\).

The tuple \((D_1, D_2, \delta_1, \delta_2)\) makes a two-way pattern structure.
Relationship between fuzzy formal concepts, pattern structures, and two-way pattern structures

Results

- It is shown that nonzero components of any crisply closed extent tuple makes an extent of an interval pattern concept, the inverse is not true in general;
- A sufficient condition for the extent of an interval pattern concept to coincide with nonzero components of a crisply closed extent tuple was obtained;
- It is shown that fuzzy formal concepts generate a two-way pattern structure. (operation $\uparrow$ defines a $\Box$).
Experiments in clustering

**Given:** 11947 objects, 5 attributes, fuzzy relation $I$.

**Output:** A cover of the set of objects by subsets of objects with “similar” attribute values.

Two methods were considered:
- A method based on interval pattern concepts;
- A method based on protofuzzy formal concepts.

Both methods construct a cover of the set of all objects by concept extents.
Define a parameter $\delta$.

1. construct all pattern concepts;
2. select those with the “width”\(\text{(maximal over components difference of attribute values)}\) not larger than $\delta$;
3. select those with maximal extents: they make the clusters;
4. delete objects forming the cluster from the context;
5. repeat the procedure until the whole set of objects is covered.
Experiments. A method based on protofuzzy formal concepts

Set a parameter $k \leq 5$.

0. Let $\gamma = 1$;

1. Decrease $\gamma$ by 0.1;

2. In $\gamma$-cut find a formal concept with the largest intent that contains not less than $k$ attributes: this concept is a cluster;

3. Delete objects from the constructed cluster;

4. Repeat the procedure while there are concepts with at least $k$ attributes in their intents). Return to step 1.
Experiments in clustering. Results

<table>
<thead>
<tr>
<th></th>
<th>$\delta$</th>
<th>method 1 (IntPS)</th>
<th>$k$</th>
<th>method 2 (protoF)</th>
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<td>5</td>
<td>0.047</td>
</tr>
</tbody>
</table>

Dunn index = \(\frac{\text{minimal distance between clusters}}{\text{maximal cluster diameter}}\)
Thank you!
References


